

A Subdivision Surface Interpolating Arbitrarily-Intersecting Network of Curves under Minimal Constraints

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Abstract—this paper presents an improved method for generating a hybrid Catmull-Clark subdivision surface that can interpolate an arbitrarily intersecting network of curves. Compared with our most recent work on this topic, the surface generation process is now simpler and altogether more intuitively appealing. Secondly, this method reduces to a bare minimum the set of constraints the surface requires to achieve interpolation. Finally, the smoothness of the surface is now considerably improved in the vicinity of extraordinary points.

Keywords—B-splines Curves; Recursive Subdivision Surfaces; Polygonal Complexes; Interpolation; X-Configurations.

I. INTRODUCTION

Curve interpolation by subdivision surfaces is a research subject that is frequently visited by the CAGD community since mid-90's following a variety of approaches. Nasri [5] initiated a technique that could generate a smooth Doo-Sabin subdivision surface interpolating a mesh of curves. This is extended in [6] to cover the interpolation of dangling quadratic B-spline curves. The control polygons of such curves are obtained by tagging edges and vertices on the initial control mesh. However, the first attempt at tackling the interpolation problem by way of Catmull-Clark subdivision surfaces is made by Schweitzer [10] and later on by Levin [4] using the Combined Subdivision Scheme.

In this context, the notion of polygonal complexes offers a general framework for solving the interpolation problem under any subdivision scheme. Very briefly, the limit of subdivision of a polygonal complex is a curve. Thus, when such a complex is embedded within a control mesh, the corresponding limit curve will naturally be interpolated by the limit surface, without the need for any further manipulation.

Our approach [1] for the interpolation of any number of intersecting curves by Catmull-Clark (CC) subdivision surfaces [3] is fundamentally based on polygonal complexes, identifying the neighborhood of the intersection points of the curves with X-Configurations. Our approach also makes use of a clear mathematical formula determining the interpolated curves in terms of the neighboring vertices on the interpolating surface.

In addition to requiring the modification of the normal Catmull-Clark subdivision coefficients near extraordinary

points, our approach was also constrained by factors that are worth noting. In fact, X-Configurations were subjected to stringent symmetry constraints seen at the time as necessary for the interpolation to take place; although the extent of that has somewhat been reduced in [2].

In this sense, the present paper is an improvement over the results reported in [2], offering an alternative hybrid subdivision scheme that altogether eliminates the need for symmetry constraints, and is at the same time more straightforward to implement and more convincing in so far as the smoothness of the limit surface is concerned.

For comparison, an alternative approach to curve interpolation is reported by Schaefer et al. [9], for constructing a Catmull-Clark subdivision surface interpolating a curve network. Their approach also integrates additional effects to enhance the smoothness of the resulting surface.

II. CUBIC SUBDIVISION OF A POLYGON

In a single subdivision step, the control polygon $P = [A_0, E_0, F_0, \dots]$ is subdivided through a polygon $Q = [A_0, e_0, m_0, n_0, p_0, q_0, \dots]$ as follows (see Figure 1):

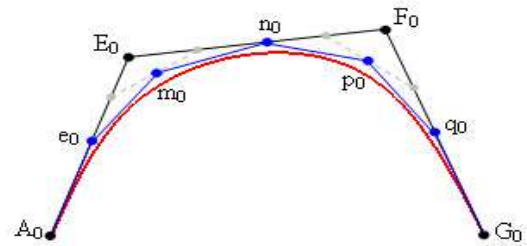


Figure 1. Cubic subdivision of a polygon

- e_0 (resp. n_0 and q_0) is the midpoint of the edge A_0E_0 (resp. E_0F_0 and F_0G_0), etc.
- m_0 (resp. p_0) is the midpoint of the edge connecting the midpoints of E_0e_0 and E_0n_0 (resp. the midpoint of the edge connecting the midpoints of F_0n_0 and F_0q_0), etc.

The extremity A_0 of the open polygon remains unchanged under this subdivision scheme. Repeating this process sufficiently often will lead to a smooth curve.

Though odd looking, the particular vertex labeling employed in this section is purposely adopted as it is destined to match the vertex labeling used in section VI with the subdivision of an X-Configuration.

III. THE CATMULL-CLARK SUBDIVISION SCHEME

A control mesh M is subdivided (see [3]) yielding another control mesh M' , as follows (see Figure 2):

- Each face F gives an F -vertex that is the average of the constituent vertices of F .
- Each inner edge E gives an E -vertex that is the average of the constituent vertices of E with the F -vertices of the neighboring faces.
- Each inner vertex V gives a V -vertex resulting from the following expression:

$$\frac{(n-2)V + \frac{R+S}{n}}{n} \quad (1)$$

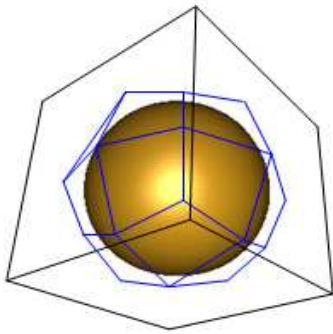


Figure 2. CC subdivision of a mesh

Where

- n is the number of faces adjacent to V
- $R = \sum_{i=1}^n V_i$ and $S = \sum_{i=1}^n V_{fi}$, where V_i is a vertex adjacent to V via an edge and V_{fi} is an F -vertex on a face f_i embodying V .

At the end of this process, each F -vertex is connected to the adjacent E -vertices and each E -vertex is connected to the adjacent V -vertices; thus forming the next mesh to be subdivided. Repeated application of the subdivision process will lead to more faces and smaller edges which, at the limit, converge on a smooth surface.

Accordingly, border edges and vertices will not contribute any new vertices. Therefore, these vertices and edges are generally kept away from the limit surface. However, these vertices and edges are sometimes incorporated into the main subdivision routine as special cases.

IV. POLYGONAL COMPLEXES

Polygonal complexes offer a general framework for solving the interpolation problem under any subdivision scheme [7, 8].

A (CC) polygonal complex is a $3 \times n$ matrix M of points representing three *parallel* polygons: top $(t_i)_i$, middle $(m_i)_i$ and bottom $(b_i)_i$, all of the same length n . These polygons are connected so as to form a sequence of pairs of rectangular faces; where each pair of faces of this sequence has a common edge and each two consecutive pairs have common respective edges (see Figure 3).

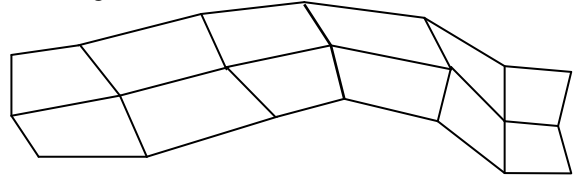


Figure 3. A (CC) Polygonal Complex

Under (CC) subdivision, while ignoring border vertices and edges at every subdivision step, a (CC) complex is gradually subdivided into a thinner and thinner complex which, at the limit, becomes a smooth curve. Thus, when such a complex is embedded within a control mesh, the corresponding limit curve will naturally be interpolated by the limit surface. Moreover, the limit of a (CC) complex M is a cubic B-spline curve whose control polygon P is given by the following formula (see [8]):

$$(1/6) \times [1 \quad 4 \quad 1] \times M \quad (2)$$

Thus, when a complex is embodied within a control mesh, its limit curve is naturally interpolated by the limit surface of this mesh. Likewise, when a complex M' is obtained from a CC complex M by replacing the mid-polygon m of M by the polygon of equation (3), the limit of M' is a B-spline curve identical to that of m (see [13]).

$$m' = (1/4) * [-1 \ 6 \ -1] * M \quad (3)$$

Thus, given a curve defined by a control polygon $(m_i)_i$, it can be transformed into a polygonal complex M through the addition to it of two more rows of points $(t_i)_i$ and $(b_i)_i$. Applying the transformation represented by equation (3), we obtain the complex M' , with the property that any mesh embodying it will in fact be interpolating the original curve defined by $(m_i)_i$.

V. X-CONFIGURATIONS

An X-Slice is a convex quadrilateral face with one of its vertices marked as a *starting* vertex and an X-Configuration is composed of $2n$ ($n \geq 2$) X-Slices, all adjacent (one to the next and the last to the first) around the same *starting* vertex (see Figure 4).

X-Configurations are designed so as to act as docking stations where (CC) complexes may be connected to their available ports. Accordingly, we classify an X-Slice as an R-Slice or as an L-Slice depending on whether it appears on the RHS or LHS of the docking polygonal complex. For example, the shaded X-Slice in figure 4 is an R-Slice.

As such, an X-Complex is a group of (CC) complexes connected via common X-Configurations. Consequently, when an X-complex is embedded in a control mesh, the network of

curves limit of the X-Complex will naturally be interpolated by the limit surface of the control mesh.

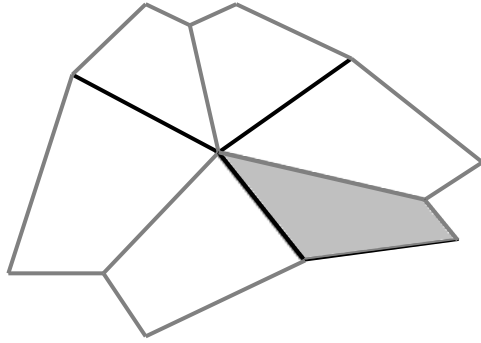


Figure 4. X-Configuration with a shaded X-Slice

VI. THE HYBRID SUBDIVISION SCHEME

The interpolation of the given curves can only be established via suitable modifications of the normal Catmull-Clark subdivision rules around the extraordinary point's centre of the X-Configurations.

An X-Slice $P_0P_1P_2P_3$ (with starting point P_0), component of an X-Configuration, will be subdivided as follows (see Figure 5):

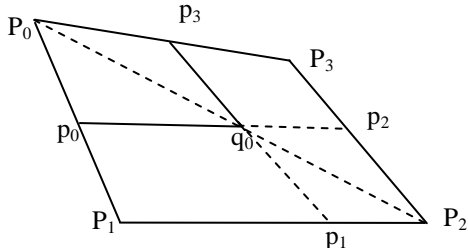


Figure 5. X-Slice Subdivision

- The F-vertex of the X-Slice is the mid-point q_0 of P_0P_3 .
- The E-Vertices of edges P_0P_1 and P_0P_3 are their respective midpoints p_0 and p_3 .
- The V-vertex of the vertex A is itself.
- Depending on whether the X-Slice $P_0P_1P_2P_3$ is an L-Slice or an R-Slice, the calculation of V-vertex corresponding to P_1 or P_3 will also be adjusted for the sake of ensuring interpolation. The specific adjustments of that will be made explicit below with reference to figure 6.

Now, with reference to figure 6 again, the only alteration that still needs to be performed is the calculation of the V-Vertex m_1 , which should become:

$$m_1 = (9E_1 + A_0 + E_2/2 + E_3/2 + F_1 + e_2 + e_3 + n_2 + n_3)/16 \quad (4)$$

All other subdivided components of the embodying mesh are determined using the normal Catmull-Clark subdivision coefficients.

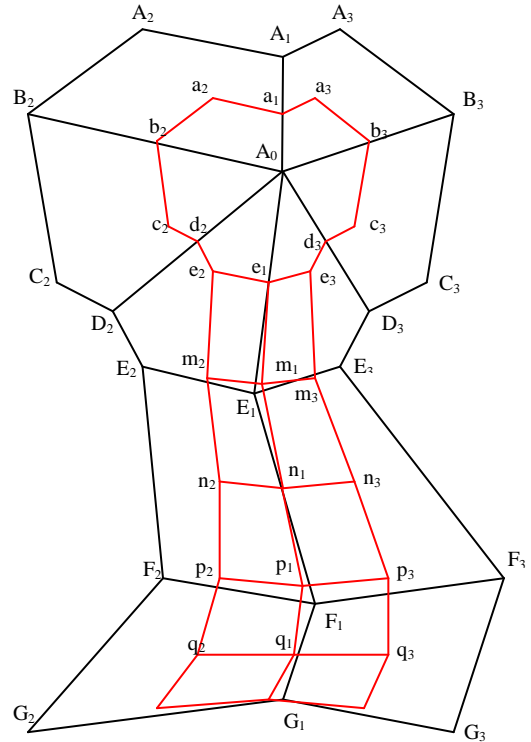


Fig. 6. One step of an X-Complex subdivision

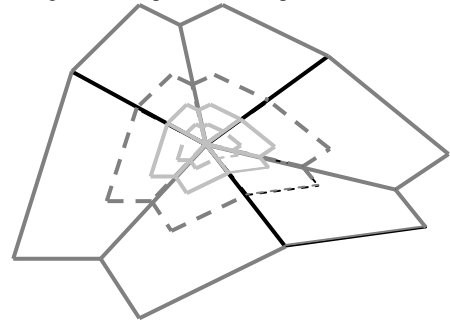


Figure 7. X-Configuration Subdivision

It is worth pointing out here that the resulting X-Slice $P_0p_1p_2p_3$ is geometrically similar to the original X-Slice $P_0P_1P_2P_3$ and so is the subdivided X-Configuration with respect to the original one (see Figure 7).

VII. MINIMAL SMOOTHNESS REQUIREMENTS

The hybrid subdivision scheme for X-Configurations introduced in the previous section is sufficient to guarantee interpolation in all situations that fit the framework depicted in this paper (see Figure 8). However, the fact that the centre vertex of the X-Configuration remains unchanged under subdivision inevitably affects the smoothness of the limit surface in those regions; i.e., in the absence of any constraints whatsoever (see figure 9, for an illustration).

In order to avoid the presence of this artifact, we only require various X-Slices of the X-Configurations to be coplanar. This is a minimum requirement that would guarantee a reasonable degree of smoothness, thus liberating the surface from more constraints.

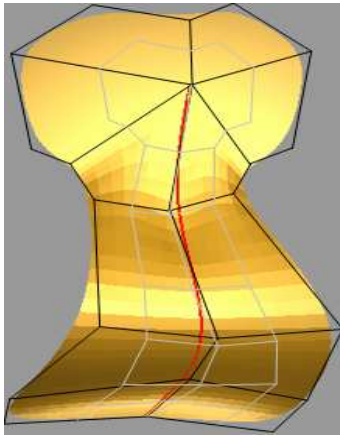


Figure 8. Planar X-Configuration

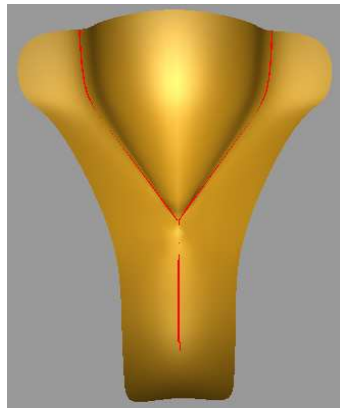
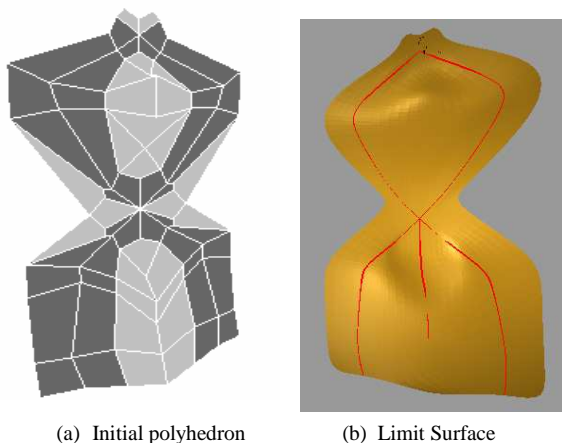


Figure 9. Non-Planar X-Configuration

VIII. SUMMARY AND FURTHER WORK

This paper presents an improved solution (over the one presented in [2]) to the interpolation of more than two curves meeting at the same point by Catmull-Clark subdivision surfaces. Figures 10 (a) and (b) illustrate the generality of the approach.



(a) Initial polyhedron

(b) Limit Surface

Figure 10. Five curves at two crossing points

As further work, we will project below how to construct a surface interpolating a set of intersecting curves.

In fact, given a set of curves intersecting at a given point, it is enough to construct an X-Complex (X) that satisfies the following conditions:

- Each of the given curves is embodied in a polygonal complex (as its middle row) of (X).
- The X-Configuration of (X) which now does not have to satisfy any the properties of symmetry whatsoever.

Having that, the repositioning of vertices as stated in equation (3) will be sufficient to guarantee that any surface embodying the resulting X-Complex will in fact be interpolating the initial curves.

ACKNOWLEDGMENT

This paper is an attempt to remedy the problem referred to in the reviewer's comment to [1] about the unfortunate narrowness of the incoming polygonal complexes to the centre of an X-Configuration.

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